

Condensed-
Matter Physics:
Inscrutably
alluring to
Physicists,
Chemists and
even
Mathematicians

Tarun Chitra

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Part 1: Symmetry-Breaking

January 13, 2013

Disclaimer

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I studied condensed-matter physics as an application of a new set of techniques known as “Geometric Transition” that arise in High-Energy Physics, Differential Geometry and Algebraic Geometry. Hence, my presentation might be quite different than what an experimental condensed-matter physicist might present. I will try to present as many experiments as I can, but by no means is this presentation complete!

What is Condensed-Matter Physics?

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- Condensed-Matter Physics is the study of matter that is in the condensed phase — Matter which is not gaseous.

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- Condensed-Matter Physics is the study of matter that is in the condensed phase — Matter which is not gaseous.
- One can define a gaseous phase in many ways, with a simple way being defined by the container that a gas is kept in. A gas can theoretically expand to an arbitrarily large volume.

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- One can define a gaseous phase in many ways, with a simple way being defined by the container that a gas is kept in. A gas can theoretically expand to an arbitrarily large volume.
- The phases were first quantified when Bernoulli (and later Maxwell and Boltzmann) realized that the three 'generic' phases of matter can be separated by their RMS velocities. Gases have the highest velocities, while solids have the lowest

Moving to the future

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- In the 21st century, the theory of solids was probed with the help of X-ray diffraction and for all intensive physics purposes, the physics of solids is considered a solved problem
- One of Albert Einstein's famous 1905 papers was entitled *On the molecular-kinetic theory of heat required by the movement of particles suspending in liquids at rest*. In this paper, Einstein makes the first connection between the Maxwell-Boltzmann Distribution (which describes the probability distribution of 'rest' velocities), Brownian motion and many-body physics

QM, for the win

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- With the advent of Quantum Mechanics in the 1920s, theorists began to tackle the physics of solids with their newfangled tools
- Theory of solids was 'finalized' by physicists such as Felix Bloch and Léon Brillouin. Enrico Fermi made important contribution to the theory of Fermionic gases

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- Liquid state, other phases of nature left out from this early-20th century boom in solid-state physics
- 1978: APS changes the name of 'Solid-State Physics' to 'Condensed-Matter Physics'

Active Areas of Research in Condensed-Matter (C-M)

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Active Areas of Research in Condensed-Matter (C-M)

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- Superfluidity and Supersolidity
- Superconductivity

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- Superfluidity and Supersolidity
- Superconductivity
- Protein Physics
- Supercooled Liquids

So... How does one formulate problems in C-M?

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- Condensed-Matter Physics involves the study of systems with *lots* of particles. Forget the three-body problem — consider the 10^{22} -body problem

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- Condensed-Matter Physics involves the study of systems with *lots* of particles. Forget the three-body problem — consider the 10^{22} -body problem
- Solid-state systems: Mathematically 'easy' because solids have a definite, static atom density that they can be approximated by a **lattice**. Lattice discretizes phase space and real space, enforces periodicity conditions.

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- Solid-state systems: Mathematically 'easy' because solids have a definite, static atom density that they can be approximated by a **lattice**. Lattice discretizes phase space and real space, enforces periodicity conditions.
- The liquid phase and various non-classical phases tend to be much more difficult to study – cannot force a lattice structure on a liquid and various non-classical phases need not be homogeneous.

All hope is not lost!

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- 1 Approximate liquids as a 'limiting' phase of solids and/or gases.
 - Classical fluid dynamics: a liquid is a solid that 'deforms' easily
 - Early QM: Fermi liquids are ideal Fermi gases (e.g n-body systems with a non-interacting, $SO(3)$ -invariant potentials) which are perturbed by an attractive interacting potential.

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- 2 Elucidate the *symmetry-breaking* that takes place in a liquid. The theory of solids is particularly nice because the system has quite a bit of symmetry; if we can figure out which symmetries of a solid still apply to liquids, then we can 'naturally' write down a Lagrangian for a liquid system

Basic Mathematical Framework: Lattices

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- Let's become a lot more formal with the definitions of the past few slides.¹

¹There are far more general definitions of a lattice such as:
'Any discrete subgroup Γ of a Lie Group G such that G/Γ has finite Haar measure.'

Basic Mathematical Framework: Lattices

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Definition

An lattice \mathcal{L} in \mathbb{R}^n is a set of \mathbb{Z} -translates of a basis $\{\vec{e}_i\}_{i=1}^n$. That is, $\mathcal{L} = \{\sum_{i=1}^n a_i \vec{e}_i \mid a_i \in \mathbb{Z}\}$. A cell is any \mathbb{Z}^n -translate of the parallelepiped spanned by $\{\vec{e}_i\}_{i=1}^n$ (the unit cell)

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- In solid-state physics, we assume that virtually all systems can locally be described by a lattice. By construction, a lattice admits a free, transitive \mathbb{Z}^n -action which means that one need only study physics in the unit cell $\text{Span}(\vec{e}_1, \dots, \vec{e}_n)$ and then use the \mathbb{Z}^n -action to translate the dynamics to any other cell. This is analogous to the setup for the NT algorithm that Anton relies on.

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- Let's recall that informal statement of Hamilton's Principle of Least Action: 'Particles are lazy, they want to take the shortest possible path while expending the least energy'

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 - We need to define the arguments of a Lagrangian: Particle's path $\mathbf{x}(t)$, velocity $\dot{\mathbf{x}}(t)$ and possibly an explicit dependence on time t and further spatial and time derivatives
 - Formal definition: A Lagrangian is a C^∞ map between an n -manifold M and a target manifold T (usually \mathbb{R} , \mathbb{C} , \mathbb{H} or \mathbb{C}^{2n} or the cotangent bundle of any symplectic manifold)

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- So now we need some way to define 'the shortest possible path that costs the least energy' in a way that is manifestly compatible with Newton's Laws
- Introducing the action \mathcal{S} . Mathematically, the action is defined as $\mathcal{S} = \int_{\mathbb{R}} \mathcal{L}(\mathbf{x}(t), \partial_{\alpha} \mathbf{x}(t), \partial^{\alpha} \partial_{\alpha} \mathbf{x}(t), \dots, t) dt$

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- Hamilton and Lagrange showed (in different ways) that by *minimizing the Lagrangian with respect to its arguments*, one could obtain a particle's path $\mathbf{x}(t)$ and velocity $\dot{\mathbf{x}}(t)$ that is consistent with what one would expect from Newtonian mechanics. This makes life a lot easier, because for a lot of problems it is much easier to 'guess' kinetic and potential energies as opposed to computing all forces

Actions and Symmetry

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Action: $\mathcal{S}[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]$.

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Action: $S[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]$.

Since $\mathbf{x}(t), \dot{\mathbf{x}}(t) \in \mathbb{R}^3$, any group that acts freely/transitively on \mathbb{R}^3 can act on the trajectories (pointwise). Representation of G : $\Lambda : G \rightarrow \text{Aut}(\mathbb{R}^3)$. G generates a **symmetry** of S if:

$$S[\mathbf{x}(t), \dot{\mathbf{x}}(t), t] = S[\Lambda(g)\mathbf{x}(t), \Lambda(g)\dot{\mathbf{x}}(t), t] \quad \forall g \in G$$

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Minimizing the action stems from a somewhat empirical consideration — in many classical physical systems, the time expectations of kinetic and potential energy were approximately equal. That is,

$$\mathbf{E}_t[T(\dot{x})] \approx \mathbf{E}_t[V(x, \dot{x})]$$

Noether's Theorem (Physics Version)

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- What about conservation laws? How do we force Lagrangian mechanics (as vaguely defined) to have conservation laws

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- What about conservation laws? How do we force Lagrangian mechanics (as vaguely defined) to have conservation laws
- Noether's Theorem (Emmy Noether, 1918): 'For every symmetry of an action, there exists a **conserved quantity** associated to the symmetry.' In classical mechanics this quantity is the *conserved momentum*:

$$P(\mathbf{x}, \dot{\mathbf{x}}) = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} K_i(\mathbf{x})$$

where $K_i(q)$ is the linear 'component' (think Taylor's theorem) of the symmetry (e.g. $\Lambda \mathbf{x} = \mathbf{x} + \epsilon \mathbf{K}(\mathbf{x})$)

Example

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Consider the Lagrangian for throwing a ball up in the air,
 $\mathcal{L} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$. This is invariant under translations of x, y , e.g. $x \mapsto x + \epsilon, y \mapsto y + \epsilon'$. These two symmetries generate two different conserved momenta (one with $K_x = 1, K_y = K_z = 0$ and the other with $K_y = 1$):

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$$P_1(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{\partial \mathcal{L}}{\partial \dot{x}} K_x + \frac{\partial \mathcal{L}}{\partial \dot{y}} K_y + \frac{\partial \mathcal{L}}{\partial \dot{z}} K_z = m\dot{x} \quad (1)$$

$$P_2(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{\partial \mathcal{L}}{\partial \dot{x}} K_x + \frac{\partial \mathcal{L}}{\partial \dot{y}} K_y + \frac{\partial \mathcal{L}}{\partial \dot{z}} K_z = m\dot{y} \quad (2)$$

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- **Hamiltonian System** (M, ω, H) : Symplectic manifold (M, ω) with Hamiltonian $H \in C^\infty(M)$. Every Hamiltonian H gives rise to a *Hamiltonian Vector Field*, $X_H = \omega^{-1}(dH)$.

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- **Infinitesimal Symmetry**: Smooth vector field $V \in \Gamma(M, TM)$ with flow $\theta : \mathcal{D} \rightarrow M$ such that $\mathbb{L}_{X_H}\omega = \omega, \theta_*H = H$.

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- **Conserved Quantity**: $f \in C^\infty(M)$ such that f is constant along any integral curve (read: trajectory) of X_H . Also defined as $f \in C^\infty(M) \ni \{H, f\} = 0$.

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Theorem

(Noether) If $f \in C^\infty(M)$ is a conserved quantity, then its Hamiltonian vector field X_f is an infinitesimal symmetry. Conversely if $H^1(M; \mathbb{R}) = 0$, then every infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity.

Symmetries of Lattices

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- Set of isometries (distance-preserving) maps = physical intuition that rotation/translation of coordinate system shouldn't affect the physics. Distance-preservation is req'd to ensure that one doesn't change an energy functional, as most energy functionals *require* a metric.

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- What are the isometries of a *cubic* unit cell? Hint: It's not a dihedral group, but it decomposes similarly (has a subgroup of reflections $\cong \mathbb{Z}_2$)

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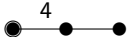
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 - Rotation Plus a Reflection (24)

Symmetries of Lattices

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- Set of isometries (distance-preserving) maps = physical intuition that rotation/translation of coordinate system shouldn't affect the physics. Distance-preservation is req'd to ensure that one doesn't change an energy functional, as most energy functionals *require* a metric.
- What are the isometries of a *cubic* unit cell? Hint: It's not a dihedral group, but it decomposes similarly (has a subgroup of reflections $\cong \mathbb{Z}_2$)
 - Proper Rotations (24): Identity rotations about an axis from center to face, rotation about an axis from the center of an edge, to an opposite edge, rotation about a body diagonal
 - Rotation Plus a Reflection (24)
 - Math: Isometry Group of Cube: $BC_3 \cong S_4 \times \mathbb{Z}_2$ is the 3rd Coxeter Group, with Coxeter-Dynkin Diagram, 

Other types of lattices

But what if they're not orthonormal? Bieberbach proved what physicists already knew: There are only a few types of crystal structures:

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Other types of lattices

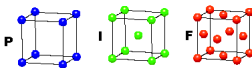
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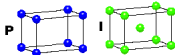
CUBIC

$$a = b = c \\ \alpha = \beta = \gamma = 90^\circ$$



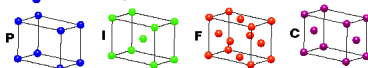
TETRAGONAL

$$a = b \neq c \\ \alpha = \beta = \gamma = 90^\circ$$



ORTHORHOMBIC

$$a \neq b \neq c \\ \alpha = \beta = \gamma = 90^\circ$$



HEXAGONAL

$$a = b \neq c \\ \alpha = \beta = 90^\circ \\ \gamma = 120^\circ$$



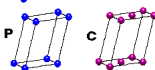
TRIGONAL

$$a = b = c \\ \alpha = \beta = \gamma \neq 90^\circ$$



MONOCLINIC

$$a \neq b \neq c \\ \alpha = \gamma = 90^\circ \\ \beta \neq 120^\circ$$



TRICLINIC

$$a \neq b \neq c \\ \alpha \neq \beta \neq \gamma \neq 90^\circ$$



4 Types of Unit Cell
P = Primitive
I = Body-Centred
F = Face-Centred
C = Side-Centred
+
7 Crystal Classes
→ 14 Bravais Lattices

Symmetry-breaking and Lattices

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Symmetry-breaking and Lattices

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- Intuitively a lattice is 'different' from an arbitrary particle because it is 'rigid'. How can symmetries help us quantify this?

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- Intuitively a lattice is 'different' from an arbitrary particle because it is 'rigid'. How can symmetries help us quantify this?
- First things first: What are the set of isometries of a single, thermally-isolated, classical particle that is sitting in a vacuum \mathbb{R}^n ?
 - Rotations — Represented in n -dimensions by the Lie group $SO(n)$
 - Translations — represented by \mathbb{R}^n , treated as Abelian group under addition

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 - Rotations — Represented in n -dimensions by the Lie group $SO(n)$
 - Translations — represented by \mathbb{R}^n , treated as Abelian group under addition
- These are *differentiable* symmetries, as the sets of rotations and translations turn out to be *Lie groups* and hence smooth manifolds (so that differentiation can be defined).

The Poincaré Group

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- This total set of symmetries of a *free particle* is known as the *Poincaré Group* $P(\mathbb{R}^n)$. However, note that $P(\mathbb{R}^3) \not\cong SO(3) \times \mathbb{R}^3$ and in general, $P(\mathbb{R}^n) \not\cong SO(n) \times \mathbb{R}^n$ for $n > 1$. To see why consider two rotations and translations $(R, \mathbf{y}), (\tilde{R}, \tilde{\mathbf{y}}), R, \tilde{R} \in SO(3), \mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^3$. A rotation plus translation acts on a vector $\mathbf{x} \in \mathbb{R}^3$ as $(R, \mathbf{y}) \cdot \mathbf{x} = R\mathbf{x} + \mathbf{y}$. Hence we have:

$$\begin{aligned}(R, \mathbf{y}) *_{\text{P}} (\tilde{R}, \tilde{\mathbf{y}}) \cdot \mathbf{x} &= (R, \mathbf{y}) \cdot (\tilde{R}\mathbf{x} + \tilde{\mathbf{y}}) \\ &= R\tilde{R}\mathbf{x} + R\tilde{\mathbf{y}} + \mathbf{y} = (R\tilde{R}, R\tilde{\mathbf{y}} + \mathbf{y}) \cdot \mathbf{x}\end{aligned}$$

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- Hence we have $P(\mathbb{R}^3) \cong \mathbb{R}^3 \rtimes_{\varphi} SO(3)$ where \rtimes_{φ} is the semi-direct product with respect to $\varphi : SO(3) \rightarrow \text{Aut}(\mathbb{R}^3)$ defined as $\varphi(R)(\mathbf{y}) = R\mathbf{y}$

Symmetry-Breaking

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- Now we can define symmetry-breaking: When dynamics force a physical system to have a smaller group (subgroup) of the static symmetries.

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- Physical system on Lattice: We can think of our system as embedded in \mathbb{R}^3 , but with a much smaller symmetry group (e.g. $BC_3 \times_{\varphi} \mathbb{Z}^n$)

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- Practical Implementation:
 - Start with a Lagrangian with a lot of symmetry \mathcal{L}_s and solve for the trajectory/dynamics
 - Perturb \mathcal{L}_s with a non-symmetric Lagrangian, \mathcal{L}_p by trying to compute the dynamics of $\mathcal{L} = \mathcal{L}_s + \epsilon \mathcal{L}_p, \epsilon > 0, \epsilon \ll 1$.
 - Likely: You will be unable to solve the Euler-Lagrange equations for this Lagrangian. Using Taylor's Theorem repeatedly (and I mean *repeatedly*) will make the new approximate solution for \mathcal{L} depend heavily on the solution for \mathcal{L}_s

Example of Symmetry-Breaking: Superfluidity

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Example of Symmetry-Breaking: Superfluidity

- Physics: Liquid ^3He has no viscosity (or creep) at $T < 3\text{mK}$. As such, there exist at least three phases of matter of ^3He . This is a *quantum* phenomena, so we need to consider spin.

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- Lagrangian density for Free Energy (*Lagrangian of Superfluid* ^3He , Phys. Rev. B, 37(7), S. Theodorakis):

$$\mathcal{L} = k_1[\Omega_{\mu\ell}^* \Omega_{\mu\ell}] + ik_2[\Omega_{\mu\ell}^* A_{\mu\ell} - \Omega_{\mu\ell} A_{\mu\ell}^*] + k_3 H_{\mu} \epsilon_{\mu\nu\sigma} A_{\nu i} \Omega_{\sigma i}^* \\ + k_3 H_{\mu} \epsilon_{\mu\nu\sigma} A_{\nu i}^* \Omega_{\sigma i} - \mathbf{F}_d - \mathbf{F}_{\text{mag}} + g \frac{\mathbf{S}^2}{2}$$

r

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r

- Symmetry-breaking: Contributions from $\mathbf{A}, \mathbf{\Omega}$ are invariant under $\text{SO}(3) \times \text{SO}(3) \times \text{U}(1)$; $\mathbf{F}_d, \mathbf{F}_{mag}$ are not. They contribute to an energetically unfavorable liquid state at low T